1. (a) Let f be a real valued function defined on the interval $(-1,1)$ such that $e^{-x}f(x) = 2 + \int_0^1 \sqrt{1+t^4}\text{dt}$, for all $x \in (-1,1)$. And let f^{-1} be the inverse function of f. Then find the value of $(f^{-1})'(2)$, where $(f^{-1})'(2)$ is the value of first order derivative of f^{-1} at 2.

[8]

(b) If $a_1, a_2, a_3, ..., a_n, ...$ are in Geometric Progression, then show that the value of det A is equal to zero, where

$$A = \begin{bmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{bmatrix}$$

[7]

2. (a) Show that $19^{93} - 13^{99}$ is divisible by 162. [10]

(b) Evaluate $\int \frac{x+3}{\sqrt{5-4x-x^2}}$ dx. [5]

3. (a) Prove that $a^5 + b^5 + c^5 > abc(ab + bc + ca)$, for all positive distinct values of a, b & c. [10]

(b) The number of people in a small country increases by 2% per year. If the population at the start of 1973 was 12,500, what is the predicted population at the start of the year 2013? [5]

4. (a) Sketch the curve of $y^2 = x^2(3-x)$ and find the area of its loop. [7]

(b) Find the centroid of the area of the loop. [8]

5. (a) If a, b are positive quantities such that $(a < b)$ and if $a_1 = \frac{a+b}{2}, b_1 = \sqrt{a_1 b}, a_2 = \frac{a_1+b_1}{2}, b_2 = \sqrt{a_2 b_1}, ..., a_n = \frac{a_{n-1}+b_{n-1}}{2}$, $b_n = \sqrt{a_n b_{n-1}}, ...$ then show that $\lim_{n\to\infty} b_n = \frac{\sqrt{b^2-a^2}}{\cos^{-1}\frac{a}{b}}$. [10]

(b) Consider $f(x) = 4x^3 - 12x$. Find the image of the interval $[-1, 3]$ under the mapping f. [5]

6. (a) For $x \geq 0$, define $f(x) = x - \sqrt{2}\sin(x)$, with $x \geq 0$ in radians.

   i. Draw the graph of f for $0 \leq x \leq 10$.

   ii. Determine the set $S = \{y : y = f(x), x \geq 0\}$. [5 + 3 = 8]

   (b) If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$, then find the set of x for which $f(x)$ increases. [7]

7. (a) Let $\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then show that $a = c = 1, b = 2$.

   [8]

   (b) Let

$$f(x) = \begin{cases} \sin(\frac{\pi x}{2}) & 0 \leq x < 1, \\ 3 - 2x & x \geq 1, \end{cases}$$

   then find the maximum of $f(x)$ if it exists. [7]

8. (a) Find the number of real solutions of the system of equations $x = \frac{2z^2}{1+z^2}, y = \frac{2x^2}{1+x^2}, z = \frac{2y^2}{1+y^2}$. [8]

   (b) Determine the ratio of height of cone of maximum volume inscribed in a sphere to its radius. [7]